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Eliminating the CPT-odd f coefficient from the Lorentz-violating standard model extension

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Abstract

The fermionic f coefficient in the Lorentz-violating standard model extension presents a puzzle. Thus far, no observable quantity that depends upon f has ever been found. We show that this is because f is actually unnecessary. It has absolutely no effects at leading order and can be completely absorbed into other coefficients of the theory by a redefinition of the field.

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In recent years, there has been a growing interest in the possibility that there could exist small Lorentz- and CPT-violating corrections to the standard model. A number of candidate theories of quantum gravity predict possible violations of these fundamental symmetries, and if any such violations were found, they would be important clues regarding the nature of Planck scale physics. An effective field theory, the standard model extension (SME) has been developed to describe all possible violations of Lorentz symmetry in quantum field theory [1, 2] and gravity [3]. The full SME is quite complicated, and so we typically restrict our attention to a field theory with only a finite number of Lorentz-violating parameters. The minimal SME contains only operators that are superficially renormalizable, and both the one-loop renormalization [4] and the stability [5] of this theory have been studied.

To date, experimental tests of Lorentz violation have included studies of matter–antimatter asymmetries for trapped charged particles [6–9] and bound state systems [10, 11], determinations of muon properties [12, 13], analyses of the behaviour of spin-polarized matter [14, 15], frequency standard comparisons [16–18], measurements of neutral meson oscillations [19–21], polarization measurements on the light from distant galaxies [22–24] and others. The results of these tests can be used to place bounds on many of the minimal SMEs Lorentz-violating coefficients. However, there are still many sectors of the theory for which there are no useful bounds at all. Since the minimal SME is used to parameterize the possible forms of Lorentz violation that might be seen in experiments, it is important to understand the structure of the model itself. In particular, we should know how many independent forms of Lorentz violation the theory can describe.

In the course of analysing these many tests of Lorentz symmetry, one puzzling fact has been observed. One set of Lorentz-violating coefficients in the Lagragian—the f terms in the fermion sector—always seem to cancel out when we calculate observable quantities. (As an immediate consequence, there are no known experimental bounds on any f.) In this paper, we shall look more closely at these coefficients. We shall show that the lack of any leading-order experimental dependences on f is actually a natural consequence of its structure. In fact, f can be completely eliminated from any theory by redefining the fields. f is reabsorbed into a different Lorentz-violating parameter, and the lowest-order f-dependent effects are of second order in the Lorentz violation. This means that f is entirely unnecessary to our description of the theory, and we may dispense with it entirely (unless using it happens to be convenient in a particular situation). These results resolve a significant puzzle, and they result in a valuable reduction in the complexity of the minimal SME.

We must begin by introducing the theory. For a model with a single species of fermion, the most general superficially renormalizable SME Lagrange density is

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\Gamma^{\mu}\partial_{\mu} - M)\psi, \tag{1}$$

where

$$\Gamma^{\mu} = \gamma^{\mu} + c^{\nu\mu}\gamma_{\nu} - d^{\nu\mu}\gamma_{\nu}\gamma_{5} + e^{\mu} + \mathrm{i}f^{\mu}\gamma_{5} + \tfrac{1}{2}g^{\lambda\nu\mu}\sigma_{\lambda\nu}. \tag{2}$$

and

$$M = m + \not{a} - \not{b}\gamma_{5} + \tfrac{1}{2}H^{\mu\nu}\sigma_{\mu\nu} + \mathrm{i}m_{5}\gamma_{5}. \tag{3}$$

Some of the coefficients in $\mathcal{L}$ are more important than others. For example, m_5 is not Lorentz violating, and it may be absorbed into the other coefficients by means of a particular field redefinition,

$$\psi' = \exp\left(-\frac{\mathrm{i}}{2}\gamma_{5}\tan^{-1}(m_{5}/m)\right)\psi, \tag{4}$$

that was already known before the introduction of the SME. This and other field redefinitions are discussed in detail in [25], although only up to leading order. In this paper, we shall be looking at effects of field redefinitions beyond leading order as well.

An a term can also be completely eliminated from the single-fermion theory, since it is essentially nothing more than a constant classical vector potential term. Removing a simply redefines the origin in momentum space—$p \to p - a$. However, if there are multiple species and flavour-changing interactions, differences in their a terms can be observable, and gravitational effects could also make a an observable quantity.

Slightly different is the antisymmetric part of c, $c^{[\nu\mu]} = c^{\nu\mu} - c^{\mu\nu}$. At leading order, the $c^{[\nu\mu]}$ terms are equivalent to a redefinition of the Dirac matrices; such a rotation in spinor space can have no physical effects. So this part of c can be eliminated with another field redefinition, but only if the $\mathcal{O}(c^2)$ terms are neglected. That the antisymmetric terms do contribute at higher order is evident from the fermions' energy–momentum relation, which is given below as equation (6).

There are also other reasons to believe that some coefficients may be more interesting than others. The e, f and g kinetic couplings appear superficially inconsistent with the coupling of the fermion field to standard model gauge fields, because they mix left- and right-chiral fields. Such terms could only arise at the electroweak breaking scale, as vacuum expectation values of nonrenormalizable operators, and so they might then be expected to be less important than the c and d terms. (However, as we shall see, there is a significant weakness in this argument. We are using the conventional Lorentz-invariant definition of the chirality operator, which might not be appropriate when the $SU(2)_L$ gauge group is coupled to Lorentz-violating matter.)

The coefficient f is similar to both a and the antisymmetric part of c Like a, f is completely unnecessary for describing the physics. However, unlike a, f has definite physical effects, although only beyond leading order. What makes f superfluous in the formalism is not that this term is unphysical, but that the effects it generates are exactly the same as those generated by another Lorentz-violating term. The more general c subsumes all the physics of an f coefficient, and f can be eliminated by absorbing it into c. At second order in f, the effects of f are indistinguishable from those of a c term:

$$c^{\nu\mu} = -\tfrac{1}{2} f^{\nu} f^{\mu}. \tag{5}$$

This situation is also similar to what occurs with m_5, as each of these terms can be entirely absorbed into other coefficients in the theory.

So far, there are no experimental bounds on f. In fact, of all the Lorentz-violating coefficients in M and Γ^{μ}, f is the only one for which there are currently no suggestions even for how it might be bounded. Typically, when searching for experimental tests of Lorentz violation, we restrict our attention to effects that appear at first order in the coefficients. Because Lorentz violation is small, any higher-order effects should be miniscule and would only be important if they caused a qualitative change in the structure of a theory. (For example, at second order in b, radiative corrections to QED could violate gauge invariance and possibly lead to a photon mass [26, 27].) The f coefficient has no physical effects at leading order, and that is precisely why its value is not constrained.

The fact that f has no leading-order effects on a theory is also related to the discrete symmetries associated with this operator. The timelike coefficient f_0 is separately odd under C, P and T. These are the same symmetries as are possessed by the spacelike parts of a and e. However, there are other discrete symmetries that distinguish these operators. The parity operator is defined as inverting all the spatial coordinates, $\vec{x} \to -\vec{x}$. However, P may be broken down into the product of three separate reflections, $P = R_1 R_2 R_3$, where R_j takes $x_j \to -x_j$ and leaves the other two coordinates unchanged. While a_j and e_j (for fixed j) are odd under R_j, they are even under the other two reflections. However, f_0 is odd under all R_j. No other minimal SME coefficient has this property. These curious symmetry properties mean that there is no other object in the theory that can combine with f to give, for example, something with the form of an $\mathcal{O}(f)$ energy shift. For similar reasons, f does not mix with any other coefficients under the action of the renormalization group [4]. (In fact, there are not even any self-renormalization terms in the one-loop β-function for f; β_f vanishes identically at leading order.)

Now to see the plausibility of our main claim, that any f term can be absorbed into c, let us look at the energy–momentum relation separately in the presence of purely spacelike c and f coefficients. The energies then are

$$E = \sqrt{m^2 + (p_k - c_{kj} p_j)(p_k - c_{kl} p_l)} \tag{6}$$

and

$$E = \sqrt{m^2 + p_j p_j + (f_k p_k)^2}. \tag{7}$$

These are actually very similar. Note that if there is only a c_{33} or an f_3, then each of these dispersion relations takes the form

$$E = \sqrt{m^2 + p_1^2 + p_2^2 + \xi p_3^2}, \tag{8}$$

where ξ is either $(1 - c_{33})^2$ or $1 + f_3^2$. This is sufficient to show that the noninteracting theories with purely spacelike c and f are equivalent. However, we obviously want to show more—that this equivalence can continue even in more complicated situations.

For definiteness, we shall continue to work with a theory containing f_3 only, demonstrating how this may be transformed into a c_{33}. This can then be generalized to cover other cases without too much difficulty, although there are some additional subtleties that arise when a timelike f is considered. The Lagrange density with f_3 only reduces to

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - \gamma_5 f_3 \partial_3 - m)\psi. \tag{9}$$

Everything is conventional, except for the matrix multiplying ∂_3. The usual $\gamma_3\partial_3$ has been replaced by $(\gamma_3 - \mathrm{i} f_3\gamma_5)\partial_3$.

The crucial observation is that $\gamma_3 - \mathrm{i} f_3\gamma_5$ anticommutes with γ^μ for $\mu \neq 3$, just as does γ_3 itself. The matrices γ_3 and $\mathrm{i}\gamma_5$ are actually completely interchangeable in the ordinary Dirac theory; they satisfy exactly the same anticommutation relations with the other Dirac matrices and possess the same normalization. So any $\gamma_3 \cos\theta - \mathrm{i}\gamma_5 \sin\theta = \gamma_3\, \mathrm{e}^{\mathrm{i}\gamma_3\gamma_5\theta}$ can actually be substituted for γ_3 in the Lorentz-invariant Lagrangian without affecting the physics.

However, $\gamma_3 - \mathrm{i} f_3\gamma_5$ does not quite have this form. Instead,

$$\gamma_3 - \mathrm{i} f_3\gamma_5 = \sqrt{1 + f_3^2}\,\gamma_3\, \mathrm{e}^{\mathrm{i}\gamma_3\gamma_5 \tan^{-1} f_3}. \tag{10}$$

The rescaling factor $\sqrt{1 + f_3^2}$ gives rise to the nontrivial c_{33} at $\mathcal{O}(f^2)$. Defining new γ-matrices by

$$\gamma'_\mu = \begin{cases} \gamma_\mu, & \mu \neq 3 \\ \gamma_3\, \mathrm{e}^{\mathrm{i}\gamma_3\gamma_5 \tan^{-1} f_3}, & \mu = 3 \end{cases} \tag{11}$$

transforms the Lagrange density into

$$\mathcal{L} = \bar{\psi}\left[\mathrm{i}\gamma'_\mu\partial^\mu - \mathrm{i}\left(\sqrt{1 + f_3^2} - 1\right)\gamma'_3\partial_3 - m\right]\psi, \tag{12}$$

that is, a Lagrange density for a theory with $c_{33} = 1 - \sqrt{1 + f_3^2}$ only.

The generalization to an arbitrary purely spacelike f is elementary:

$$\gamma'_j = \gamma_j\, \mathrm{e}^{\mathrm{i} f_k\gamma_k\gamma_5 G(f_l f_l)} = \mathrm{e}^{-\frac{\mathrm{i}}{2} f_k\gamma_k\gamma_5 G(f_l f_l)}\gamma_j\, \mathrm{e}^{\frac{\mathrm{i}}{2} f_k\gamma_k\gamma_5 G(f_l f_l)}, \tag{13}$$

where $G(x) = \frac{1}{\sqrt{x}} \tan^{-1}\sqrt{x}$. Note that $G(x)$ is analytic around $x = 0$, so the arguments of the exponents depend analytically on the components of f. It is a trivial matter to recast this as a redefinition of the field, rather than the Dirac matrices. The exponentials commute with γ_0, so the field redefinition is simply

$$\psi' = \mathrm{e}^{-\frac{\mathrm{i}}{2} f_k\gamma_k\gamma_5 G(f_l f_l)}\psi \tag{14}$$

$$\bar{\psi}' = \bar{\psi}\, \mathrm{e}^{\frac{\mathrm{i}}{2} f_k\gamma_k\gamma_5 G(f_l f_l)}, \tag{15}$$

and the Lagrangian for ψ' contains only a c term.

What is left is to deal with timelike f_0 terms. The correct generalization of (14) is obvious:

$$\psi' = \mathrm{e}^{\frac{\mathrm{i}}{2} f^\mu\gamma_\mu\gamma_5 G(-f^2)}\psi, \tag{16}$$

but there are some slight complications associated with the timelike case. For a purely timelike f, with f_0 only, the field redefinitions become

$$\psi' = \mathrm{e}^{\frac{\mathrm{i}}{2}\gamma_0\gamma_5 \tanh^{-1} f_0}\psi \tag{17}$$

$$\bar{\psi}' = \bar{\psi}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\gamma_0\gamma_5 \tanh^{-1} f_0}. \tag{18}$$

This converts f_0 into a $c_{00} = \sqrt{1 - f_0^2} - 1 \approx -\frac{1}{2} f_0^2$, and here we see the subtlety. For spacelike f, the transformation could be effected for an arbitrary negative f^2; however, when the Lorentz-violating coefficient is timelike, the field redefinition is only possible if $f^2 < 1$. Larger values give rise to an imaginary c and so a Lagrangian that is not Hermitian. This is not surprising, for if $f^2 > 1$, then the square of the matrix multiplying ∂_0 in the Lagrangian becomes negative, and so the time evolution can no longer be unitary. In that case, the entire theory becomes inconsistent.

Since the field redefinition (16) works to eliminate f in both the purely spacelike and purely timelike cases, it can be shown to hold for an arbitrary f, simply by performing the relevant calculations in a boosted frame. The c that is generated by the transformation is

$$c^{\nu\mu} = \frac{f^\nu f^\mu}{f^2}\left(\sqrt{1-f^2}-1\right). \tag{19}$$

This may be further generalized to the case in which the initial Lagrangian has both c and f terms. In that case, f may still be absorbed into a modification of c, and again there are no $\mathcal{O}(f)$ terms. However, the resulting expression is rather cumbersome, and it is uninteresting practically. What is important is that the leading contribution that f^μ makes to $c^{\nu\mu}$ is unchanged and remains equal to $-\frac{1}{2}f^\nu f^\mu$. Infinitesimally, the field redefinition (16) we have found is identical with that presented in [25], where it was pointed out that this would eliminate f from the free Lagrangian at leading order. Also as discussed in [25], this kind of transformation will generally reshuffle any other Lorentz-violating coefficients that are present in the theory. For example, if the theory prior to the elimination of f contains a b term, then the field redefinition will generate a $H^{\mu\nu}$ proportional to $(f^\mu b^\nu - b^\mu f^\nu)$. Likewise, if the theory initially contains an a, the field redefinition will generate a m_5 proportional to $f^\mu a_\mu$; fortunately however, if there exists a concordant frame, in which all the Lorentz-violating coefficients are small, it is indeed possible to eliminate both f and m_5 from the theory, using slightly more involved field redefinitions.

The expression (19) is indeterminate for lightlike f, but the limiting value as $f^2 \to 0$, $c^{\nu\mu} = -\frac{1}{2}f^\nu f^\mu$, is correct at $f^2 = 0$. This can be verified, for example, using light cone coordinates. Otherwise, (19) holds formally for all other $f^2 < 1$ (although, as with $G(-f^2)$, the power series expansion about $f^2 = 0$ fails for $f^2 < -1$). However, the larger f behaviour of the theory is fairly uninteresting, for two reasons. First, for all observed particles, Lorentz violation is small. Second, if a theory did contain a large f or large c, there would be causality violations at a low energy scale, invalidating the description in terms of effective field theory anyway [5].

We expect the coefficients describing any physical Lorentz violation to be of characteristic size $\mathcal{O}(m/M_P)$, where m is a typical mass scale (i.e. in the $\sim$ 1–100 GeV range), and M_P is some very large scale, possibly the Planck scale. (The actual values of the coefficients will vary by type and by particle species, so m/M_P is only a very rough estimate of the size of the Lorentz violation.) Typically, the description of the physics in terms of effective field theory breaks down at energies comparable to M_P. Additional higher-dimension operators must be introduced at that scale if properties such as causality are to be preserved. However, the c term is an exception to this. Because c and the Lorentz-invariant kinetic term possess the same basic structure, there is mixing between them, and the c-modified theory fails at the lower scale $\sqrt{mM_P}$.

To see this, observe that the velocity in the presence of a purely spacelike c (chosen for simplicity) is

$$v_k = \frac{1}{E}(p_k - c_{kj}p_j - c_{jk}p_j + c_{jk}c_{jl}p_l). \tag{20}$$

This can become superluminal when $|\vec{p}|/E \approx 1 - |c|$, where $|c|$ is a characteristic size for the Lorentz-violating coefficients. For ultrarelativistic particles, for which $1 - |\vec{v}| \ll 1$, the

Lorentz factor is roughly $\gamma \approx 1/\sqrt{2(1-|\vec{v}|)}$. This sets the scale of γ at which new physics must enter: $\gamma_{\max} \sim 1/\sqrt{|c|}$. This corresponds to an energy scale $E_{\max} \sim \sqrt{mM_P}$.

So it seems that there may be a conflict between the version of the theory containing f, which breaks down at the higher scale M_P, and the version with c, which could fail at a lower scale. However, this problem is alleviated by the fact that the c term related to f is actually of $\mathcal{O}(f^2)$, and so its natural size is $\mathcal{O}\left(m^2\big/M_P^2\right)$. When f is converted into c, the energy at which things break down is just the geometric mean between m and $M_P^2\big/m$, and this is exactly M_P. So the scale of new physics is defined consistently in either framework.

The fact that f can be absorbed into c is also related to the leading-order triviality of $c^{[\nu\mu]}$. There are five independent mutually anticommuting 4×4 matrices, which may be arranged in any way we like as γ^μ and γ_5 (with appropriate factors of i). The elimination of f fixes the definition of γ_5 and removes four of the ten degrees of freedom associated with changes to the representations of the Dirac matrices. However, there are still six unphysical degrees of freedom contained in c. The quantity $g^{\nu\mu}+c^{\nu\mu}$ defines a bilinear form that connects p_μ and γ_ν in the action. This bilinear form contains sixteen free parameters. However, the physics in a theory with a c-type modification ultimately depends only on the energy–momentum relation, which can be expressed as a bilinear form that connects p with itself. So only the symmetric part of this second bilinear form is physical, and this amounts to only ten physical parameters. The six parameters that are unphysical are exactly those that correspond to the $SO(3,1)$ transformations that change the representation of γ^μ. At leading order, these transformations are represented precisely by $c^{[\nu\mu]}$; however, at higher orders, the algebraic characterization of which parts of c are trivial becomes more complicated.

However, there is a fairly simple geometrical characterization of which parts of c are actually physical. With c as the only form of Lorentz violation, the fermionic energy–momentum relation takes the general form

$$C^{\nu\mu}C_\nu{}^\rho p_\mu p_\rho - m^2 = 0, \tag{21}$$

in terms of $C^{\nu\mu} = g^{\nu\mu}+c^{\nu\mu}$. We shall work in a fixed frame and consider $C^{\nu\mu} = (C^\nu)^\mu$ as a 'vector of vectors'. The inner index (ν) is coupled to the specific Dirac representation, while the outer index (μ) may be seen simply as a parameter. It is then clear from (21) that only quantities formed from inner products of the (C^ν) vectors can have physical consequences. The outer indices parameterize ten of these inner products; these are the ten physical parameters and precisely the ten constants that define the bilinear form in (21). So in essence, only the magnitudes and relative orientations of these vectors are physical. The overall orientation of the cluster of vectors has no physical consequences, and $SO(3,1)$ rotations of the entire cluster parameterize the six unphysical parameters.

To leading order, the four f^μ coefficients are just the angles that parameterize a rotation in spinor space. It is then natural that f is not renormalized at this order; any radiative corrections to f would actually be quantum corrections to the Dirac matrix representation. Conversely, the choice of Dirac matrices should not affect the renormalization of any of the theory's other parameters. So there are no $\mathcal{O}(f)$ terms in any of the β-functions of Lorentz-violating QED [4], for example. At second order in f, on the other hand, there are radiative corrections to c.

We have discussed a field redefinition that eliminates f from the Lagrangian. However, another type of field redefinition is often used when the theory is considered in the Hamiltonian framework (e.g. in [6, 7, 10]). If Γ^0 is invertible, then the Dirac equation may be recast in the Schrödinger-like form

$$\mathrm{i}\partial_0\psi = (\Gamma^0)^{-1}(\mathrm{i}\vec{\Gamma}\cdot\vec{\nabla}+m)\psi. \tag{22}$$

However, the operator appearing on the right-hand side of (22) will not generally be Hermitian, because there were nonstandard time derivative terms in the original Lagrangian. Using a field redefinition $\psi = (\gamma^0\Gamma^0)^{-1/2}\psi'$, we may transform (22) into a new equation with a Hermitian Hamiltonian, provided that $\gamma^0\Gamma^0$ is positive definite [5, 28].

We shall now examine how these alternate field redefinitions behave in the presence of a Lorentz-violating f only, so that $\Gamma^0 = \gamma^0 + \mathrm{i} f^0\gamma_5$. Invertibility of this matrix only requires that $f_0^2 \neq 1$. However, $\gamma^0\Gamma^0$ will not be positive definite unless the stronger condition $f_0^2 < 1$ is met. This condition for the existence of the field redefinition is also stronger than the condition $f^2 < 1$ that we encountered when looking at transformations of the Lagrangian—a fact which should be unsurprising. In order to have a well-defined Hamiltonian formulation, we must also be able to define the theory properly via its Lagrangian. However, since the Hamiltonian method chooses a particular frame, it can be less advantageous. The cost of choosing a reference frame in which $|f^0|$ is greater than its minimum value $\sqrt{\max(f^2, 0)}$ is that we must have $f_0^2 < 1$, rather than merely $f^2 < 1$, in order to define the theory. In essence, by examining the theory in an inopportune frame, we are not making the best use of the spacelike Lorentz-violating coefficients, which could be used to improve the theory's behaviour. Finally, we point out that $f_0^2 < 1$, as it is not a Lorentz-invariant condition, could be violated, even for small f^2, in a highly boosted frame; this again illustrates that the problems in defining the Hamiltonian are associated with choosing a poor choice of frame when quantizing the theory.

While f is unnecessary for our description of the SME fermion sector, it is still possible that it might prove convenient to use this parameter in specific situations. Effects that depend on c in a particular fashion might be more simply expressed in terms of f. For example, some of the most stringent bounds on c for the electron come from observations of synchrotron radiation from the Crab nebula [29]. The spacelike coefficients so bounded take the form $c_{jk}\hat{e}_j\hat{e}_k$, where $\hat{e}$ is a unit vector. So this constraint is on exactly that part of c that has the form $c_{jk} = \pm v_j v_k$ for some vector $\vec{v}$. This suggests that a formulation in terms of f might be more succinct. Yet unfortunately, the bound in [29] is one-sided. A positive $c_{jk}\hat{e}_j\hat{e}_k$ leads to a maximum electron velocity in the direction of $\hat{e}$, and that is a phenomenon with readily measurable effects. However, a negative $c_{jk}\hat{e}_j\hat{e}_k$ does not lead to a maximum velocity, so no $c_{jk} = -\frac{1}{2}f_j f_k$ is excluded by this measurement—although, if a bound on a negative $c_{jk}\hat{e}_j\hat{e}_k$ was available, it would immediately translate into a bound on $|f_j\hat{e}_j|$ in a formulation of the theory involving f.

Nothing that we have discussed will change if the conserved vector current is coupled to a gauge field. The derivative ∂_μ is simply replaced by a covariant derivative D_μ. The inclusion of the vector potential does not affect the field redefinition in any way. This may seem a trivial observation, but there are situations where similar conclusions do not hold. A b term may be eliminated from a massless noninteracting theory by a different kind of field redefinition, $\psi' = \mathrm{e}^{-\mathrm{i}\gamma_5 b^\mu x_\mu}\psi$. This corresponds to separate translations of momentum space for the left- and right-handed fermions. While an Abelian vector coupling does not appear to mix the two helicities, it is well known that chiral symmetry is broken at $\mathcal{O}(\hbar)$ by the anomaly. So, even though it looks like this field redefinition should eliminate b entirely from the physics, that coefficient can still contribute to quantum corrections.

Nor do we expect a coupling to gravity to affect our ability to eliminate f. The field redefinition that transforms away this coefficient is really just a change in the basis used for the Dirac matrices. A gravitational interaction is not coupled in any way to the specific Dirac matrices used to define a theory, so a rotation in spinor space is still allowed, even in curved spacetime. This is in contrast to what happens with the a term, which cannot generally be removed when there is a nontrivial spacetime background. The reason is that the field redefinition $\psi' = \mathrm{e}^{\mathrm{i}a^\mu x_\mu}\psi$ which removes a is x-dependent, and this dependence interacts nontrivially with the covariant derivative. However, since f is removed by an x-independent field redefinition, there are no analogous problems associated with its elimination.

Finally, we must address the issue of couplings to chiral gauge theories. As previously noted, an f term—but not a c term—will mix left- and right-chiral fermion fields. This appears to contradict the fact that a f may be converted into a c. However, the reasoning that leads to the contradiction is actually based on an erroneous assumption. We have assumed that the chiral projectors that appear in the Lagrangian should have the form $\frac{1\pm\gamma_5}{2}$. However, while the chiral current $\bar{\psi}\Gamma^{\mu}\gamma_5\psi$ is not conserved if $M = 0$ and $f \neq 0$, there is another conserved current, $\bar{\psi}\Gamma^{\mu}\gamma_5'\psi$, with $\gamma_5' = \gamma_5 + \mathcal{O}(f)$. In fact, the necessary γ_5' is simply $-\mathrm{i}\gamma_0'\gamma_1'\gamma_2'\gamma_3' = \mathrm{e}^{\frac{1}{2}f^{\mu}\gamma_{\mu}\gamma_5 G(-f^2)}\gamma_5\,\mathrm{e}^{-\frac{1}{2}f^{\mu}\gamma_{\mu}\gamma_5 G(-f^2)}$. The modified chiral current is conserved, because γ_5' anticommutes with all Γ^{μ}. So the theory can consistently be coupled to an $SU(2)_L$ gauge group, provided the left-chiral projector used is actually $\frac{1-\gamma_5'}{2}$. (In a similar vein, there is a modified CPT operator under which the theory with f is even, like the theory with c.)

However, the existence of a modified chirality operator is a special property of the theory containing f. Such an operator does not exist for general Γ^{μ}. The question of whether such an operator exists is closely tied to the relationship between e and f, which we shall now briefly discuss. For $m = m_5 = 0$, the energy–momentum relations have the same form for theories with either e or f as the sole Lorentz-violating coefficients. (More generally, e gives the same dispersion relation as theory with particular a, c and modified m coefficients.) Since the energy–momentum relations and particle statistics completely define a noninteracting quantum field theory, this means that the theories with either solely e or solely f (and no masses) describe the same physics. However, in a theory with e^0 as its only form of Lorentz violation, there is no matrix $\gamma_5 + \mathcal{O}(e)$ that anticommutes with $\Gamma^0 = \gamma^0 + e^0$. Yet actually there is a field redefinition that will convert an e term into an f term in precisely the $m = m_5 = f = 0$ case:

$$\psi' = \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\gamma_5}\psi = \frac{1}{\sqrt{2}}(1 - \mathrm{i}\gamma_5)\psi. \tag{23}$$

The terms with e and f have the same Dirac matrix structures as m and m_5, respectively, so it might seem obvious that in the massless theory, we can eliminate e in favour of f, just as m could be eliminated in favour of m_5. However, because the full term containing e involves a derivative, the necessary field redefinition is nonlocal. The only exception to this is if $f = 0$ initially, so that the argument of the inverse tangent in the analogue of (4) is singular; the resulting transformation is exactly (23). The necessary field redefinition does not vanish as $e \to 0$, and so the theory's modified chirality operator does not have the form $\gamma_5 + \mathcal{O}(e)$. If the theory initially contains both nonzero e and f, then we could still attempt to construct a new chirality operator via a field redefinition. The resulting operator would formally obey the correct Clifford algebra anticommutation relations, but it would actually be nonlocal. In the presence of interactions with additional spacetime-dependent fields, the nonlocal field redefinitions will not work to eliminate e from the theory, because ∂^{μ} and x_{μ} do not commute. Therefore, unless either $m = m_5 = f = 0$ or $m = m_5 = e = 0$, the explicit breaking of chiral symmetry is real and unavoidable. Moreover, if $m \neq 0$, then e and f are definitely not equivalent. There can be physical effects of $\mathcal{O}(e)$ involving gravity, while f can always be eliminated in favour of c that is $\mathcal{O}(f^2)$.

So we have seen that the f coupling is really quite special. While it has no effects at linear order, it is not trivial in general. However, there are no unique phenomena associated with this form of Lorentz violation. The f coefficient can be removed from the theory by a spacetime-independent field redefinition, which replaces a pure f term with a c term, provided that $f^2 < 1$. However, only small values of f^2 are really interesting, both because they represent the only possible physical regime and because there are causality violations at an unacceptably low scale if f^2 is comparable to unity. For small f, the f^μ coefficient is equivalent to $c^{\nu\mu} \approx -\frac{1}{2} f^\nu f^\mu$.

The field redefinition that eliminates f in favour of c is compatible with vector and chiral gauge couplings, as well as a coupling to gravity. This implies that f is actually a completely extraneous parameter in the SME. For each fermion species, it may be transformed away. So further consideration of the f coefficients is unnecessary, and this represents an important simplification to the structure of Lorentz-violating effective field theory.

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References

[1] Colladay D and Kostelecký V A 1997 *Phys. Rev.* D **55** 6760
[2] Colladay D and Kostelecký V A 1998 *Phys. Rev.* D **58** 116002
[3] Kostelecký V A 2004 *Phys. Rev.* D **69** 105009
[4] Kostelecký V A, Lane C D and Pickering A G M 2002 *Phys. Rev.* D **65** 056006
[5] Kostelecký V A and Lehnert R 2001 *Phys. Rev.* D **63** 065008
[6] Bluhm R, Kostelecký V A and Russell N 1997 *Phys. Rev. Lett.* **79** 1432
[7] Bluhm R, Kostelecký V A and Russell N 1998 *Phys. Rev.* D **57** 3932
[8] Gabrielse G, Khabbaz A, Hall D S, Heimann C, Kalinowsky H and Jhe W 1999 *Phys. Rev. Lett.* **82** 3198
[9] Dehmelt H, Mittleman R, Van Dyck R S Jr and Schwinberg P 1999 *Phys. Rev. Lett.* **83** 4694
[10] Bluhm R, Kostelecký V A and Russell N 1999 *Phys. Rev. Lett.* **82** 2254
[11] Phillips D F, Humphrey M A, Mattison E M, Stoner R E, Vessot R F C and Walsworth R L 2001 *Phys. Rev.* D **63** 111101(R)
[12] Bluhm R, Kostelecký V A and Lane C D 2000 *Phys. Rev. Lett.* **84** 1098
[13] Hughes V W, Perdekamp M G, Kawall D, Liu W, Jungmann K and zu Putlitz G 2001 *Phys. Rev. Lett.* **87** 111804
[14] Bluhm R and Kostelecký V A 2000 *Phys. Rev. Lett.* **84** 1381
[15] Heckel B *et al* 1999 *Elementary Particles and Gravitation* ed B N Kursonoglu *et al* (New York: Plenum)
[16] Berglund C J, Hunter L R, Krause D Jr, Prigge E O, Ronfeldt M S and Lamoreaux S K 1995 *Phys. Rev. Lett.* **75** 1879
[17] Kostelecký V A and Lane C D 1999 *Phys. Rev.* D **60** 116010
[18] Bear D, Stoner R E, Walsworth R L, Kostelecký V A and Lane C D 2000 *Phys. Rev. Lett.* **85** 5038
[19] Kostelecký V A 2000 *Phys. Rev.* D **61** 016002
[20] Hsiung Y B 2000 *Nucl. Phys. Proc. Suppl.* **86** 312
[21] Abe K *et al* 2001 *Phys. Rev. Lett.* **86** 3228
[22] Carroll S M, Field G B and Jackiw R 1990 *Phys. Rev.* D **41** 1231
[23] Carroll S M and Field G B 1997 *Phys. Rev. Lett.* **79** 2394
[24] Kostelecký V A and Mewes M 2001 *Phys. Rev. Lett.* **87** 251304
[25] Colladay D and McDonald P 2002 *J. Math. Phys.* **43** 3554
[26] Altschul B 2004 *Phys. Rev.* D **69** 125009
[27] Altschul B 2004 *Phys. Rev.* D **70** 101701(R)
[28] Lehnert R 2004 *J. Math. Phys.* **45** 3399
[29] Altschul B 2005 *Phys. Rev.* D **72** 085003